THE STABILITY OF PLANE FLOW FOR A NON-NEWTONIAN FLUID OBEYING A RHEOLOGICAL POWER LAW

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With respect to small perturbations, we examine the stability of a steady flow (with a gradient) of a non-Newtonian fluid obeying a rheological power law in a flat channel. We have found the neutral stability curves for various values of the exponent n in the rheological law.

In this paper we will investigate the stability of a steady plane flow with a gradient for fluids obeying a rheological power law, for which the relationship between the deviator of the stress tensor s_{ij} and the strain-rate tensor f_{ij} (the rheological law) is written [1] in the form

$$s_{ij} = 2k_n \omega^{n-1} f_{ij} \quad (n > 0,\ i, j = 1, 2, 3), \tag{1}$$

where $\omega = \sqrt{2f_{ij}f_{ij}}$. On the basis of the adopted terminology, media with $n > 1$ are referred to as dilatational fluids, while those with $n < 1$ are known as pseudoplastic. The case $n = 1$ corresponds to a Newtonian fluid.

From the equation of motion for the medium, written in the absence of body forces,

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial s_{ij}}{\partial x_j} \tag{2}$$

(ρ is the density of the medium; p is the pressure; v_i is the component of the velocity vector) for a steady flow in a plane channel under the action of a constant pressure gradient in the direction of the axis $x_1 \equiv x$ ($v_1 = U$, $v_2 = v_3 = 0$) with consideration of the boundary conditions we can find the profile of the dimensionless velocity [2] in the form

$$U(y) = 1 - |y|^{\frac{n+1}{n}}, \tag{3}$$

with the axis $x_2 \equiv y$ perpendicular to the channel wall; in making the transition to the dimensionless quantities, we have taken the maximum velocity at the center of the channel for the case in which $y = 0$ as the characteristic velocity; we have taken the half-width of the channel as the characteristic dimension.

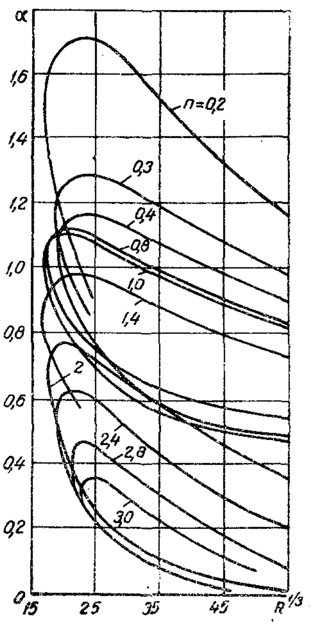

Fig. 1. Neutral stability curves.

The stability of flow (3) is studied in relation to small two-dimensional perturbations in the velocities u' and v' along the x- and y-axes, respectively. The equations of motion and continuity are linearized in the usual manner [3]. If we introduce the stream function for the perturbations

$$u' = \frac{\partial \Psi}{\partial y}; \qquad v' = -\frac{\partial \Psi}{\partial x} \tag{4}$$

and seek the solution for Ψ in the form

$$\Psi(x, y, t) = \psi(y) \exp[i\alpha(x - ct)], \tag{5}$$

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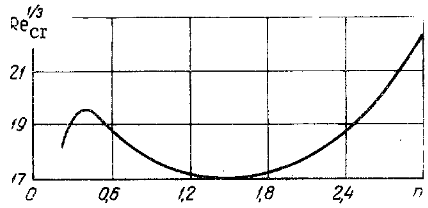

Fig. 2. Critical value of the generalized Reynolds number as a function of the exponent n in the rheological law.

where αc is a complex dimensionless frequency of perturbations, we can derive the generalized Orr–Sommerfeld equation for fluids with a rheological power law. For regions of flow $-1 \le y \le 0$, in which $dU/dy < 0$, the generalized Orr–Sommerfeld equation has the form

$$\begin{aligned}[(U-c)(D^2-\alpha^2)-(D^2U)]\psi = \frac{(DU)^{n-3}}{i\alpha \mathrm{Re}}\{(DU)^2 n (D^2-\alpha^2)^2 \\ + (n-1)\{2n(DU)(D^2U)D^3 + [4\alpha^2(DU)^2 + n(DU)(D^3U) \\ + n(n+2)(D^2U)^2]D^2 + 2(n-2)\alpha^2(DU)(D^2U)D \\ + \alpha^2 n[(DU)(D^3U) + (n-2)(D^3U)^2]\}\}\psi, \end{aligned} \tag{6}$$

where $D \equiv d/dy$, and $\mathrm{Re} = \rho U_{char}^{2-n} L_{char}^{n}/k_n$ is the generalized Reynolds number for power-law fluids. With $n = 1$, Eq. (6) changes into the Orr–Sommerfeld equation [3].

The boundary conditions for the function ψ are set at the half-width of the channel at the points $y_1 = -1$ and $y_2 = 0$, with the latter condition understood as the limit. For even perturbations, which are the most dangerous from the standpoint of flow stability, the boundary conditions are the following:

$$\psi(y_1) = D\psi(y_1) = D\psi(y_2) = D^3\psi(y_2) = 0. \tag{7}$$

If ψ is given by the asymptotic expansion

$$\psi(y) = \sum_{s=0}^{\infty} \frac{\psi^{(s)}(y)}{(\alpha\,\mathrm{Re})^s}, \tag{8}$$

the first pair of independent solutions of (6) is found from

$$(U-c)(D^2-\alpha^2)\psi - (D^2U)\psi = 0, \tag{9}$$

which is the equation of the zeroth approximation of $\psi(y)$ in (8). The solutions of (9) can be found in the form of power series in $y - y_c$, where y_c is the point at which $U(y_c) = c$:

$$\begin{aligned}\psi_1^{(0)} &= (y-y_c)\sum_{k=0}^{\infty} a_k (y-y_c)^k, \\ \psi_2^{(0)} &= \psi_1^{(0)} \ln(y-y_c)\frac{D^2U(y_c)}{DU(y_c)} + \sum_{k=0}^{\infty} b_k (y-y_c)^k,\end{aligned} \tag{10}$$

where

$$a_0 = b_0 = 1;\; a_1 = b_1 = \frac{1}{2ny_c};\;\; a_2 = \frac{\alpha^2}{6} + \frac{1-n}{6n^2y_c^2};\;\; b_2 = \frac{\alpha^2}{2} + \frac{1+2n}{4n^2y_c^2};$$

$$a_3 = \frac{\alpha^2}{18ny_c} + \frac{(1-n)(1-2n)}{24n^3y_c^3};\;\; b_3 = \frac{\alpha^2}{36ny_c} + \frac{4n^2-4n-3}{24n^3y_c^3};$$

$$a_4 = \frac{\alpha^4}{120} + \frac{\alpha^2}{60n}\left(\frac{11}{12n}-1\right)\frac{1}{y_c^2} + \frac{(1-n)(1-2n)(1-3n)}{120n^4y_c^4};$$

$$b_4 = \frac{\alpha^4}{24} - \frac{\alpha^2(13+36n)}{432n^2y_c^2} - \frac{6-n-20n^2+12n^3}{144n^4y_c^4};$$

$$a_5 = \frac{23\alpha^4}{10800ny_c} - \frac{(180n^2-242n+71)\alpha^2}{21600n^3y_c^3} + \frac{(1-n)(1-2n)(1-3n)(1-4n)}{720n^5y_c^5};$$

$$b_5 = -\frac{\alpha^4}{3600ny_c} - \frac{\alpha^2(64+147n-180n^2)}{5400n^3y_c^3} + \frac{144n^4-300n^3+90n^2+80n-29}{2880n^5y_c^5};$$

$$a_6 = \frac{\alpha^6}{5040} - \frac{\alpha^4(270n-233)}{453600n^2y_c^2} - \frac{\alpha^2(780n^3-1270n^2+597n-86)}{151200n^4y_c^4} + \frac{(1-n)(1-2n)(1-3n)(1-4n)(1-5n)}{5040n^6y_c^6};$$

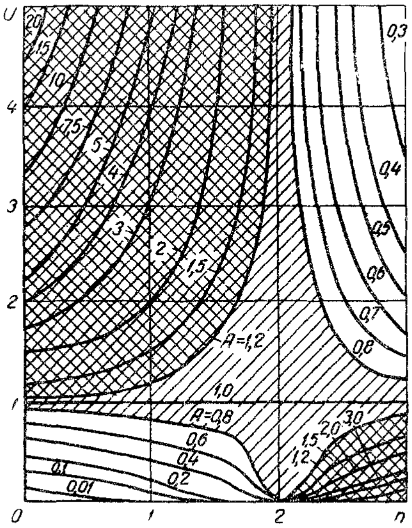

Fig. 3. Critical value of the maximum velocity in the channel as a function of n.

$$b_6 = \frac{\alpha^6}{720} - \frac{\alpha^4 (675n + 178)}{162000 n^2 y_c^2} - \frac{\alpha^2 (6300n^3 - 7880n^2 + 388n + 1106)}{324000 n^4 y_c^4} - \frac{20160n^5 - 48048n^4 + 25508n^3 + 7742n^2 - 7462n + 1155}{604800 n^6 y_c^6}.$$

Another pair of independent particular solutions of (6) is found in the form

$$\psi = \exp\left(\int g\,dy\right); \quad g = \sum_{m=0}^{\infty} (\alpha\,\mathrm{Re})^{\frac{1-m}{2}} g_m. \tag{11}$$

Substitution of (11) into (6) enables us to determine

$$g_0 = \pm\sqrt{\frac{i(U-c)}{n(DU)^{n-1}}}; \quad g_1 = -\frac{5DU}{4(U-c)} + \frac{(n-1)D^2U}{4DU}, \tag{12}$$

as a result of which we can find

$$\psi_{3,4} = (U-c)^{-\frac{5}{4}} (DU)^{\frac{n-1}{4}} \exp\left[\mp \int_{y_c}^{y} \sqrt{\frac{i\alpha \mathrm{Re}(U-c)}{n(DU)^{n-1}}}\,dy\right]. \tag{13}$$

Solutions ψ_3 and ψ_4 near $y = y_c$ are found directly from (6) on introduction of the new variable

$$\eta = \frac{y - y_c}{\varepsilon}; \quad \varepsilon = (\alpha \mathrm{Re})^{-\frac{1}{3}}. \tag{14}$$

If we seek the solution $\psi(y) \equiv \chi(\eta)$ in the form of a series in powers of ε

$$\chi(\eta) = \sum_{k=0}^{\infty} \varepsilon^k \chi^{(k)}, \tag{15}$$

after equating the coefficients for identical powers of ε we find

$$\chi_1^{(0)} = \eta, \quad \chi_2^{(0)} = 1,$$

$$\chi_3^{(0)} = \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} \sqrt{\eta}\, H_{1/3}^{(1)}\left[\frac{2}{3}(ia\eta)^{\frac{3}{2}}\right] d\eta, \tag{16}$$

$$\chi_4^{(0)} = \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} \sqrt{\eta}\, H_{1/3}^{(2)}\left[\frac{2}{3}(ia\eta)^{\frac{3}{2}}\right] d\eta,$$

where $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$ are Hankel functions, and

$$a = \sqrt[3]{\frac{[DU(y_c)]^{2-n}}{n}}. \tag{17}$$

The asymptotic Hankel function enables us to identify χ_1 and χ_2 with the solutions ψ_1 and ψ_2, and χ_3 and χ_4 with the solution ψ_3 and ψ_4, as well as to determine the required branch in the circumvention of y_c

$$-\frac{7\pi}{6} < \arg(y - y_c) < \frac{\pi}{6}. \tag{18}$$

The condition of nontriviality for the general solution of (6), according to the usual procedure [3], leads to the secular equation which, after evaluating the terms in order of magnitude, is written in the form

$$\frac{D\psi_3(y_1)}{\psi_3(y_1)} = \frac{\begin{vmatrix} D\psi_1(y_1) & D\psi_2(y_1) \\ D\psi_1(y_2) & D\psi_2(y_2) \end{vmatrix}}{\begin{vmatrix} \psi_1(y_1) & \psi_2(y_1) \\ D\psi_1(y_2) & D\psi_2(y_2) \end{vmatrix}}. \tag{19}$$

The left-hand member of (19) is expressed in terms of the tabulated Tietjens function, while the right-hand member is calculated by means of the found solutions for (10). The solution of the transcendental equation

(19) by the Tollmien [3] method leads to neutral curves which separate the stability region from the non-stability region at the (α, Re) plane.

Figure 1 shows the neutral curves calculated for the values of n = 0.1, 0.3, 0.4, 0.8, 1.0, 1.4, 2.0, 2.8, and 3.0. Figure 2 shows $Re_{cr}^{1/3}$ as a function of n. As we can see from the curve, the value of the critical generalized Reynolds number over a wide range of variation in n changes only slightly. Nevertheless, the stability losses in the laminar channel flow of a fluid obeying a rheological power law will be realized at various values of n for various values of the critical velocity U_{cr}. If we introduce the notation $A = k_n Re_{cr}^{(n)} / \rho L^n$, for determined values of A, using Re_{cr} as a function of n, on the (U_{cr}, n) plane we can construct a family of lines separating the stability and instability regions. The cross-hatched areas in Fig. 3 correspond to the velocity values at which we have a loss in laminar-flow stability when A = 1.2 and 0.8.

NOTATION

s_{ij}	is the stress-tensor deviator;
f_{ij}	is the strain-rate tensor;
ω	is the intensity of the strain-rate tensor;
k_n, n	are rheological constants of the medium;
U	is the velocity of the steady flow;
u', v'	are components of the velocity perturbations;
Ψ	is the stream function for the perturbations;
ψ	is the amplitude of the perturbations in the stream function;
α	is a real dimensionless wave number;
Re	is the generalized Reynolds number for the power-law fluids;
D	is the differentiation operator;
$\psi_1, \psi_2, \psi_3, \psi_4$ ($\chi_1, \chi_2, \chi_3, \chi_4$)	are independent particular solutions of the Orr–Sommerfeld equation.

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